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ABRIDGED TABLES OF HYPERBOLIC FUNCTIONS

BY

F. E. PERNOT

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ABRIDGED TABLES OF HYPERBOLIC FUNCTIONS.

BY

F. E. PERNOT

In the calculation of the operating characteristics of long transmission circuits the most convenient and direct solution is afforded by the use of hyperbolic functions of complex variables. For lines consisting of large conductors in which the losses are small it is desirable to make computations to the degree of accuracy afforded by six-place tables of logarithms. Also, in such cases, the argument to which the hyperbolic functions are taken from the tables is small; for power circuits usually below 0.5.

The most convenient table of these functions is that of the Smithsonian Institution, in which are tabulated both the logarithms and natural values for arguments from zero to 6.0, five decimal places being given. A table of Gudermannian functions is also appended, which makes it possible to find the values of functions (hyperbolic) to six places; not, however, without involving a double interpolation, first to the Gudermannian and then from a table of trigonometric functions.

For small arguments the hyperbolic cosine is not varying rapidly, hence easy interpolations are assured. The hyperbolic sine, on the other hand, is varying with the same rapidity as the argument, and therefore the interpolations for log sinh x are cumbersome unless the tabular interval is very small. These considerations are immediately seen from the series expressing the two functions:

$$\left.\begin{aligned} \cosh x &= 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \frac{x^6}{\lfloor 6} + \ldots\ldots \\ \sinh x &= x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \frac{x^7}{\lfloor 7} + \ldots\ldots \end{aligned}\right\} \quad (1)$$

In all cases the hyperbolic tangent is most easily derived from sinh x and cosh x by

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad (2)$$

The series for sinh x in (1) can be written

$$\sinh x = x\,(1 + \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} + \frac{x^6}{\lfloor 7} + \ldots .) \quad (3)$$

which immediately suggests the advisability of tabulating values of the quantity in brackets,

$$y = 1 + \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} + \frac{x^6}{\lfloor 7} + \ldots . \quad (4)$$

for this quantity is seen to have an even smaller rate of change than cosh x. The value of the hyperbolic sine is then given by

$$\left.\begin{aligned} \sinh x &= xy \\ y &= \frac{\sinh x}{x} \end{aligned}\right\} (5)$$

Since extended computations are most conveniently done by logarithms, the tabulation was made of log y. Log x naturally being at hand, the value of log sinh x is immediately found by simple addition of log x and log y, and, further, it is found without having to make any inconvenient interpolations. A glance at the values of log sinh x as tabulated in any table will immediately impress one with the impracticability of interpolating directly for log sinh x for values of x between 0 and 0.10, in which region a large proportion of the values of x fall for the particular work above referred to. This scheme of tabulating the ratio of a function of a variable to the variable itself is the same as is used in the "S and T" tables found in Bremiker's logarithm tables, where for small angles the ratios of the trigonometric sine and tangent to the angle are given.

CONSTRUCTION OF TABLES

The following table contains values to base 10 of $\log \frac{\sinh x}{x}$ and log cosh x, together with the differences to be used in interpolating. The arguments progress in steps of 0.005 from zero to 0.600, giving 121 entries. Logarithms are given to seven places.

The first column, $\log_{10} \frac{\sinh x}{x}$, was computed by first evaluating the expression for y in (4) and then taking out log y from tables. For the small values of x used, this series is very convergent, hence the labor involved was not excessive. The evaluation of the series was made for alternate entries, and the intermediate values obtained by interpolation.

Using this value of $\log \frac{\sinh x}{x}$ the value of $\log \frac{1}{\sinh^2 x}$ was formed. Using this as argument in Zech's table of addition logarithms, the value of $\log_{10} \cosh^2 x$ is immediately obtained, from the relation $\cosh^2 x = \sinh^2 x + 1$. Interpolations in the addition logarithm tables were made to the nearest even number in order that the resulting value of log cosh x might appear to the nearest unit in the last place.

The complete set of values was checked by differences, and in a few cases the last unit was changed by one in order to give uniformity in the second differences, which in both tabulations are practically constant.

In addition to the check by differences, every tenth entry was checked independently by calculating directly the values of

$$y = \frac{\sinh x}{x} = \frac{\epsilon^x - \epsilon^{-x}}{2x} \text{ and } \cosh x = \frac{\epsilon^x + \epsilon^{-x}}{2},$$

using the tabulated values of the exponentials as given to nine decimals in tables of the exponential function by J. W. L. Glaisher, F. R. S., published in the *Transactions of the Cambridge Philosophical Society*, vol. XIII. From the above values, the logarithm to seven places was taken from tables, which agreed with the previously tabulated values to the nearest unit of the last place, except in two or three cases where a difference of one was noted.

To facilitate interpolation, the values of the first derivatives of the function multiplied by the tabular interval ($\omega = 0.005$) are tabulated in units of the last place in the tabulated seven-place logarithm of the function. The second differences are also tabulated.

For the first three columns:

$$\left.\begin{aligned} f(x) &= \log_{10}\frac{\sinh x}{x} = \log_{10} y \\ \omega f'(x) &= \omega\frac{d}{dx}\log_{10}\frac{\sinh x}{x} = \omega\frac{d}{dx}\log_{10} y \\ &= \omega\frac{d}{dx}\log_{10}\left(1 + \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} + \ldots\right) \\ &= 2\omega M\frac{x}{y}\left(\frac{1}{\lfloor 3} + \frac{2x^2}{\lfloor 5} + \frac{3x^4}{\lfloor 7} + \frac{4x^6}{\lfloor 9} + \ldots\right) \end{aligned}\right\} \quad (6)$$

M = modulus of the common logarithm system = 0.43429448.

This series is very convergent, and was used in the form for the value of $\omega = 0.005$,

$$\omega f'(x) = 0.01M\frac{x}{y}\left(\frac{1}{\lfloor 3} + \frac{2x^2}{\lfloor 5} + \frac{3x^4}{\lfloor 7} + \ldots\right) \quad (7)$$

Of course the above value was multiplied by 10^7 to reduce to units of the last place in the tabulation of log y. Values of log y to be used in the computation were taken directly from the table.

Δ_2 is the average of the differences in $\omega f'(x)$ immediately preceding and following a tabular value of log y.

For the last three columns:

$$\left.\begin{aligned} f(x) &= \log_{10} \cosh x \\ \omega f'(x) &= \omega \frac{d}{dx} \log_{10} \cosh x \\ &= \omega M \frac{\sinh x}{\cosh x} \end{aligned}\right\} \quad (8)$$

Thus

$$\omega f'(x) = 0.005M \tanh x \quad (9)$$

Δ_2 is the same as defined for the previous case.

It is to be noted that the second differences, or accelerations if x be considered equicrescent, are practically constant. Easy and accurate interpolations are thus assured, even to seven decimals and for tabular intervals in the argument as great as 0.005, which is five times as great an interval as is used in the Smithsonian Tables for the functions tabulated to five decimals only.

TO INTERPOLATE USING SECOND DIFFERENCES

Let it be required to determine the value of $f(x)$ for an argument

$$x = x_0 \pm a$$

x_0 is the value of the argument nearest that of x. a, then, is the distance over which the interpolation has to be made, and $\frac{a}{\omega} = n =$ the fraction of the tabular interval. Using the second differences, we have

$$f(x) = f(x_0 \pm n\omega) = f(x_0) \pm n\, \omega f'(x) + \frac{n^2}{2}\Delta_2 \quad (10)$$

$f(x_0)$ is the value of the function corresponding to the argument x_0.

Illustration:

Required,	log sinh	0.1163740
	log cosh	0.1163740
	$x =$	0.1163740
	$x_0 =$	0.115
	$a =$	+0.0013740
	$n = \frac{a}{.005} =$	+ 0.27480

	for sinh x	for cosh x
$f(x_0)$	= 0.0009569	0.0028655
$n\, \omega f'(x)$	= 228.6	683.2
$\frac{1}{2} n^2 \Delta_2$	= 1.4	4.1
$f(x)$	= 0.0009799	0.0029342
log x	= 9.0658560—10	
log sinh x	= 9.0668359—10	log cosh x = 0.0029342

log tanh x = 9.0639017—10

LOGARITHMS OF HYPERBOLIC FUNCTIONS OF A REAL VARIABLE

From $x=0$ to $x=0.600$

x	$\log_{10}\frac{\sinh x}{x}$	$\omega f'(x)$	Δ_2	$\log_{10}\cosh x$	$\omega f'(x)$	Δ_2
0.000	0.0000000	000.0	36.2	0.0000000	000.0	108.6
.005	.0000018	36.2	36.2	.0000054	108.6	108.6
.010	.0000072	72.4	36.2	.0000217	217.1	108.6
.015	.0000162	108.6	36.2	.0000489	325.7	108.6
.020	.0000289	144.8	36.2	.0000869	434.2	108.5
.025	.0000452	180.9	36.2	.0001357	542.8	108.5
.030	.0000651	217.1	36.2	.0001954	651.2	108.5
.035	.0000886	253.3	36.2	.0002659	759.7	108.4
.040	.0001158	289.5	36.2	.0003473	868.1	108.4
.045	.0001466	325.7	36.2	.0004396	976.5	108.4
.050	.0001809	361.8	36.2	.0005426	1084.8	108.3
.055	.0002189	398.0	36.2	.0006565	1193.1	108.2
.060	.0002605	434.2	36.2	.0007813	1301.3	108.2
.065	.0003057	470.4	36.2	.0009168	1409.5	108.1
.070	.0003546	506.5	36.2	.0010632	1517.5	108.0
.075	.0004071	542.7	36.2	.0012203	1625.6	108.0
.080	.0004632	578.8	36.1	.0013882	1733.5	107.9
.085	.0005229	615.0	36.1	.0015670	1841.3	107.8
.090	.0005862	651.1	36.1	.0017565	1949.1	107.7
.095	.0006531	687.2	36.1	.0019568	2056.7	107.6
.100	.0007236	723.3	36.1	.0021679	2164.3	107.5
.105	.0007977	759.4	36.1	.0023897	2271.7	107.4
.110	.0008755	795.6	36.1	.0026222	2379.0	107.3
.115	.0009569	831.7	36.1	.0028655	2486.2	107.2
.120	.0010418	867.8	36.1	.0031194	2593.3	107.0
.125	.0011304	903.8	36.1	.0033841	2700.3	106.9
.130	.0012226	939.9	36.1	.0036595	2807.1	106.8
.135	.0013184	976.0	36.0	.0039456	2913.8	106.6
.140	.0014178	1012.0	36.0	.0042423	3020.4	106.5
.145	.0015208	1048.1	36.0	.0045496	3126.8	106.3
.150	.0016274	1084.1	36.0	.0048676	3233.0	106.2
.155	.0017376	1120.1	36.0	.0051962	3339.1	106.0
.160	.0018514	1156.1	36.0	.0055354	3445.0	105.8
.165	.0019688	1192.1	36.0	.0058852	3550.8	105.7
.170	.0020899	1228.1	36.0	.0062456	3656.3	105.5
.175	.0022145	1264.1	36.0	.0066165	3761.8	105.3
.180	.0023427	1300.1	36.0	.0069979	3867.0	105.1
.185	.0024745	1336.0	36.0	.0073899	3972.0	104.9
.190	.0026099	1372.0	36.0	.0077923	4076.9	104.7
.195	.0027489	1407.9	35.9	.0082052	4181.5	104.5
.200	.0028915	1443.8	35.9	.0086286	4286.0	104.3

x	$\log_{10} \frac{\sinh x}{x}$	$\omega f'(x)$	Δ_2	$\log_{10} \cosh x$	$\omega f'(x)$	Δ_2
.200	.0028915	1443.8	35.9	.0086286	4286.0	104.3
.205	.0030377	1479.7	35.9	.0090624	4390.2	104.1
.210	.0031874	1515.6	35.9	.0095066	4494.2	103.9
.215	.0033407	1551.4	35.9	.0099612	4598.0	103.7
.220	.0034977	1587.3	35.8	.0104262	4701.6	103.5
.225	.0036582	1623.1	35.8	.0109015	4805.0	103.3
.230	.0038223	1658.9	35.8	.0113872	4908.1	103.0
.235	.0039900	1694.7	35.8	.0118832	5011.1	102.8
.240	.0041612	1730.5	35.8	.0123894	5113.7	102.5
.245	.0043361	1766.3	35.8	.0129059	5216.2	102.3
.250	.0045145	1802.1	35.7	.0134326	5318.3	102.1
.255	.0046965	1837.8	35.7	.0139696	5420.3	101.8
.260	.0048821	1873.5	35.7	.0145167	5522.0	101.6
.265	.0050712	1909.2	35.7	.0150739	5623.4	101.3
.270	.0052639	1944.9	35.7	.0156413	5724.5	101.0
.275	.0054602	1980.5	35.6	.0162188	5825.4	100.8
.280	.0056600	2016.2	35.6	.0168064	5926.1	100.5
.285	.0058634	2051.8	35.6	.0174040	6026.4	100.2
.290	.0060704	2087.4	35.6	.0180117	6126.5	99.9
.295	.0062809	2123.0	35.6	.0186293	6226.3	99.7
.300	.0064950	2158.5	35.5	.0192569	6325.8	99.4
.305	.0067126	2194.0	35.5	.0198945	6425.0	99.1
.310	.0069338	2229.6	35.5	.0205419	6523.9	98.8
.315	.0071586	2265.1	35.5	.0211993	6622.5	98.5
.320	.0073869	2300.6	35.5	.0218665	6720.9	98.2
.325	.0076187	2336.0	35.4	.0225435	6818.9	97.9
.330	.0078541	2371.5	35.4	.0232302	6916.6	97.6
.335	.0080930	2406.9	35.4	.0239267	7014.0	97.2
.340	.0083354	2442.2	35.4	.0246330	7111.1	96.9
.345	.0085814	2477.6	35.3	.0253490	7207.8	96.6
.350	.0088309	2512.9	35.3	.0260746	7304.3	96.3
.355	.0090839	2548.2	35.3	.0268098	7400.4	96.0
.360	.0093405	2583.5	35.3	.0275546	7496.2	95.6
.365	.0096006	2618.8	35.2	.0283090	7591.7	95.3
.370	.0098643	2654.0	35.2	.0290730	7686.8	95.0
.375	.0101315	2689.2	35.2	.0298464	7781.6	94.6
.380	.0104022	2724.4	35.2	.0306293	7876.1	94.3
.385	.0106764	2759.5	35.1	.0314216	7970.2	93.9
.390	.0109541	2794.7	35.1	.0322233	8064.0	93.6
.395	.0112353	2829.8	35.1	.0330344	8157.4	93.3
.400	.0115201	2864.9	35.1	.0338548	8250.5	92.9
.405	.0118083	2899.9	35.0	.0346845	8343.2	92.6
.410	.0121000	2934.9	35.0	.0355234	8435.6	92.2
.415	.0123952	2969.9	35.0	.0363716	8527.6	91.8
.420	.0126940	3004.9	35.0	.0372289	8619.2	91.5
.425	0.0129963	3039.8	34.9	.0380954	8710.5	91.1

x	$\log_{10}\dfrac{\sinh x}{x}$	$\omega f'(x)$	Δ_2	$\log_{10}\cosh x$	$\omega f'(x)$	Δ_2
.425	0.0129963	3039.8	34.9	.0380954	8710.5	91.1
.430	.0133020	3074.7	34.9	.0389710	8801.4	90.7
.435	.0136112	3109.6	34.9	.0398557	8892.0	90.4
.440	.0139239	3144.5	34.8	.0407494	8982.2	90.0
.445	0.142400	3179.3	34.8	.0416521	9072.0	89.6
.450	.0145597	3214.1	34.8	.0425638	9161.4	89.3
.455	.0148828	3248.8	34.7	.0434844	9250.5	88.9
.460	.0152095	3283.5	34.7	.0444139	9339.2	88.5
.465	.0155396	3318.2	34.7	.0453522	9427.5	88.1
.470	.0158732	3352.9	34.6	.0462993	9515.4	87.7
.475	.0162102	3387.5	34.6	.0472552	9602.9	87.3
.480	.0165507	3422.1	34.6	.0482199	9690.1	86.9
.485	.0168946	3456.7	34.6	.0491932	9776.8	86.6
.490	.0172420	3491.2	34.5	.0501752	9863.2	86.2
.495	.0175929	3525.7	34.5	.0511659	9949.2	85.8
.500	.0179472	3560.2	34.5	.0521651	10034.8	85.4
.505	.0183049	3594.6	34.4	.0531728	10119.9	85.0
.510	.0186661	3629.0	34.4	.0541890	10204.7	84.6
.515	.0190307	3663.4	34.3	.0552137	10289.1	84.2
.520	.0193988	3697.7	34.3	.0562468	10373.1	83.8
.525	.0197703	3732.0	34.3	.0572883	10456.7	83.4
.530	.0201452	3766.3	34.2	.0583382	10539.9	83.0
.535	.0205235	3800.5	34.2	.0593963	10622.7	82.6
.540	.0209053	3834.7	34.2	.0604627	10705.1	82.2
.545	.0212905	3868.8	34.1	.0615373	10787.1	81.8
.550	.0216791	3903.0	34.1	.0626201	10868.6	81.4
.555	.0220711	3937.0	34.1	.0637111	10949.8	81.0
.560	.0224665	3971.1	34.0	.0648101	11030.6	80.6
.565	.0228653	4005.1	34.0	.0659171	11111.0	80.2
.570	.0232675	4039.1	34.0	.0670322	11190.9	79.7
.575	.0236731	4073.0	33.9	.0681553	11270.4	79.3
.580	.0240821	4106.9	33.9	.0692863	11349.5	78.9
.585	.0244945	4140.8	33.8	.0704252	11428.3	78.5
.590	.0249103	4174.6	33.8	.0715720	11506.5	78.1
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.600	.0257519	4242.2	33.7	.0738888	11661.9	77.3

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